The “Complementary Filter” that is not a Complementary Filter

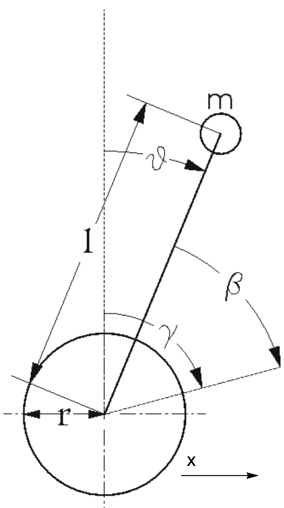
According to leading internet authorities, the complementary filter for tilt can be implemented as...
angle = $\int \text{gyro}\, dt$ + acc

where gyro and acc represent gyro and accelerometer readings. Acc is in units of g, the gravitational constant. The thinking is that the first term gives low frequency information, the second term high frequency information and the resulting angle is sufficiently accurate to be useful. The filter is coded (not by me) as...
angle = 0.98 *(angle+gyro*dt) + 0.02*acc

See the problem? For dt = .010, you can rewrite this is as...
angle = .98*angle + (.98*gyro + 2*acc)*dt

Forgetting about coding errors, an accelerometer used for tilt measurement actually measures...
acc = θ *g - $\frac{d^2x}{dt^2}$

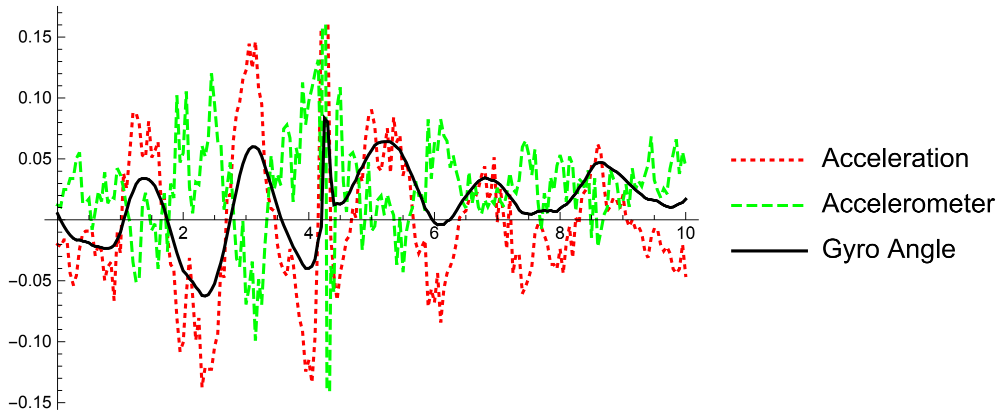
So is the second term negligible? It turns out no. And it turns out the two components are actually of opposite sign. You can measure all these quantities on a balancing robot. The three quantities to be measured and recorded are the accelerometer, the integrated gyro and the lateral acceleration.



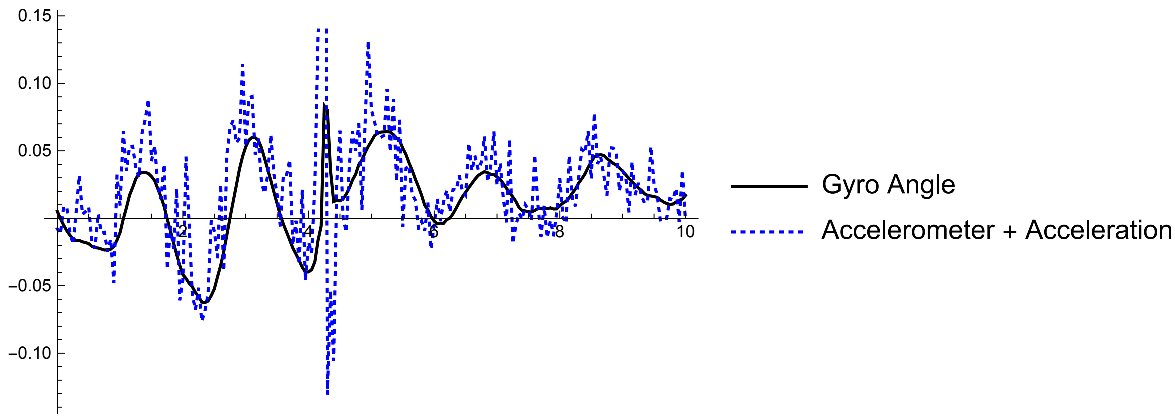
Assume the lateral dimension, x, is positive to the right. The robot wheels turn with angle β as shown. Then

$\frac{dx^2}{dt^2} = r\,\frac{d\beta^2}{dt^2}$ and lateral acceleration can be measured using wheel encoders.

Question: When measured using the accelerometer, will the tilt and acceleration components be in phase or out of phase? Wrong. They will be out of phase. Imagine you are in a boxcar on the robot and the tilt, θ, increases and the boxcar leans to the right. Now, to keep balance β will increase moving the boxcar to the right and the acceleration will throw you to the left, in the opposite direction of the boxcar leaning. The lateral acceleration is to the right, the lean is to the right, but the accelerometer measures lateral acceleration opposite to the leaning. Here is 10 seconds worth of data.



Notice that the acceleration as measured with the wheel encoders has greater magnitude than the accelerometer. In theory, adding the accelerometer with the out of phase acceleration cancels out the acceleration and the result is the tilt angle. The data looks pretty good.



So, if the "complementary filter" mostly measures linear acceleration how does it balance? To answer, look at two controllers that ***I have used to balance***. This controller uses gyro data, ϕ, and wheel encoders, β, as inputs to calculate the motor input, σ ...

$$\sigma(s) = \frac{a_0 + a_1 * s}{s^2}\,\phi(s) + \frac{c_0 + c_1 * s}{s}\,\beta(s)$$

The wheel angular velocity, $\frac{d\beta}{dt}$, is approximately proportional to the motor input, σ. Let the constant gain be G. Then G*σ(s)=s*β(s) can be substituted in the above equation to yield...

$$\sigma(s) = \frac{a_0 + a_1 * s}{s^2}\,\phi(s) + \frac{c_0 + c_1 * s}{s^2}\,\sigma(s)$$

$$\frac{\sigma(s)}{\phi(s)} = \frac{a_0 + a_1 * s}{s^2 - G * c_0 - G * c_1 * s}$$

You can see that the denominator has positive real roots and is unstable. Used in conjunction with the unstable inverted pendulum the result is stable. The controller has positive feedback. It uses either using β or σ to provide positive feedback. I suspect that using the accelerometer as a "complementary filter" does the same thing.